Summer Work Packet for MPH Math Classes

Students going into
AP Calculus AB
Sept. 2020

Name: ______________________

This packet is designed to help students stay current with their math skills and be prepared for an ADVANCED PLACEMENT course.

Each math class expects a certain level of number sense, algebra sense, and graph sense in order to be successful in the course. These problems need to be completed by September 11th. Be sure to show all work. We will check this assignment in class and will have quizzes on these topics during the first quarter.

If you have any questions, please email me at dmeehan@mphschool.org.

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The TI 84+ calculator is good for use in AP Calculus. It does everything you are allowed to use it for on the AP Exam.

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Functions & Graphs **AP CALCULUS AB** **Name ____________________**

Put all graphs on graph paper. Be sure to label the scale on each axis.

1. Are the functions the same? Explain.

   a. $f(x) = \frac{1}{\sqrt{x-3}}$ and $g(x) = \frac{\sqrt{x-3}}{x-3}$

   b. $f(x) = \sqrt{x-3}$ and $g(x) = \frac{x-3}{\sqrt{x-3}}$

2. Graph $f(x) = |x|$.

   a. Graph the **slope** of $f(x) = |x|$.

   b. Write a piecewise function that is equivalent to *f(x)*.

   c. Graph $g(x) = \frac{x}{|x|}, x \neq 0$.

   d. Write a piecewise function that is equivalent to *g(x)* that does not use absolute value.

   e. Graph $h(x) = x|x|$.

   f. Write a piecewise function that is equivalent to *h(x)* that does not use absolute value.

3. Graph $j(x) = [\![x]\!]$.

   a. As x approaches –1 from the left, what does *j(x)* approach? ($x \to -1^-$, $j(x) \to$ _____)

   b. As x approaches –1 from the right, what does *j(x)* approach? ($x \to -1^+$, $j(x) \to$ _____)

   c. As x approaches 0 from the left, what does *j(x)* approach? ($x \to 0^-$, $j(x) \to$ _____)

   d. As x approaches 0 from the right, what does *j(x)* approach? ($x \to 0^+$, $j(x) \to$ _____)

   e. As x approaches 1 from the left, what does *j(x)* approach? ($x \to 1^-$, $j(x) \to$ _____)

   f. As x approaches 1 from the right, what does *j(x)* approach? ($x \to 1^+$, $j(x) \to$ _____)

   g. How would describe the results?

4. Graph $k(x) = \frac{2x^2 - 3x}{x^2 - 25}$. State the domain. Discuss its vertical and horizontal asymptotes.

5. Graph $l(x) = \frac{x\sqrt{x^2-6x+9}}{x-3}$. State the domain. Discuss its vertical and horizontal asymptotes.

6. Write an equation of a function that has a removable discontinuity (hole) at the point (3, 4).

7. Graph *m(x)* = cos(x) in the interval [–2π, 2π].

    a. For what values of x is *k(x)* increasing?

    b. For what values of x is *k(x)* decreasing?

8. Graph *n(x)* = sec(x) in the interval [–2π, 2π].

    a. For what values of x is *m(x)* concave up (like a cup)?

    b. For what values of x is *k(x)* concave down (like a frown)?

9. Given *p(x)* = $a(x - r_1)(x - r_2)(x - r_3)(x - r_4)\dots(x - r_n)$ and $n \in Z$. Discuss the end behavior (as $x \to \infty$ and $x \to -\infty$) of the function. Be sure to consider odd and even values of *n,* and the role of the coefficient, *a*.

10. Discuss the end behavior (as $x \to \infty$ and $x \to -\infty$) of the function. Be sure to consider different values of *b*.

a. Given $q(x) = b^x$, $x \in R;\ b > 0$.

b. Given $r(x) = \log_b(x)$, $x \in R;\ b > 0$.

11. On your **calculator**, graph $s(x) = x + 3\cos(x)$ in the window $[-4\pi, 4\pi]$ by $[-20, 20]$. Discuss the interesting aspects of the function.

12. Let $f(x) = 2^x\ and\ g(x) = \sqrt{x}$.

a. Write the domain of f(x) and g(x).

b. Compare the domains of $f(g(x))\ and\ g(f(x))$.

13. Let $f(x) = |x|$ *and* $g(x) = \sin(x)$.

a. Graph $f(g(x))$ *and* $g(f(x))$.

b. Write a brief explanation of why the graph is what it is.

14. If *f(–x) = –f(x)*, then the function is ODD, symmetric to the origin, and both A(x, y) and B(–x, –y) are on the graph. If *f(–x) = f(x)*, then the function is EVEN, symmetric to the y-axis, and both A(x, y) and B(–x, y) are on the graph.

On your **calculator**, graph the following functions and determine whether each is odd, even, or neither.

a. y = sin(x) ODD EVEN NEITHER

b. y = cos(x) ODD EVEN NEITHER

c. y = tan(x)) ODD EVEN NEITHER

d. y = cot(x) ODD EVEN NEITHER

e. y = sec(x)) ODD EVEN NEITHER

f. y = csc(x)) ODD EVEN NEITHER

15. Definition: If $\boldsymbol{f(g(x)) = g(f(x)) = x}$, then $f(x)$ *and* $g(x)$ are INVERSE functions.

Their graphs are symmetrical to the line y = x.

$g(x) = f^{-1}(x)$.

If (a, b) is a point on *f(x)*, then (b, a) is a point on $f^{-1}(x)$.

Use this definition to determine, algebraically, if the functions are inverses.

a. f(x) = ½ x + 4; g(x) = 2x – 8

b. f(x) = x^2; g(x) = $\sqrt{x}$

16. Find the inverse function of $f(x) = \frac{1}{x}$.

Rational Functions **AP CALCULUS AB** **Name ____________________**

GIVEN: $p(x) = x^2 - 1$ and $q(x) = x^3 + 1$

1. Graph $W(x) = \frac{p(x)}{q(x)}$ on graph paper. (Calculator window: Z4 to start)

2. Graph $Z(x) = \frac{q(x)}{p(x)}$ on graph paper. (Calculator window: Z4 to start)

3. Adjust your window as necessary. Check the table values. For each function, find the domain, range, end behavior model and end behavior asymptote, vertical asymptote(s), hole(s), local/absolute maximum(s) and minimum(s) and real root(s). Reference the last activity we did from your binder.

W(x)	Z(x)
Domain:	Domain:
Range:	Range:
Vertical Asymptote(s):	Vertical Asymptote(s):
Holes:	Holes:
End Behavior Model:	End Behavior Model:
End Behavior Asymptote:	End Behavior Asymptote:
Relative Maximum(s):	Relative Maximum(s):
Relative Minimum(s):	Relative Minimum(s):
Real Roots:	Real Roots:

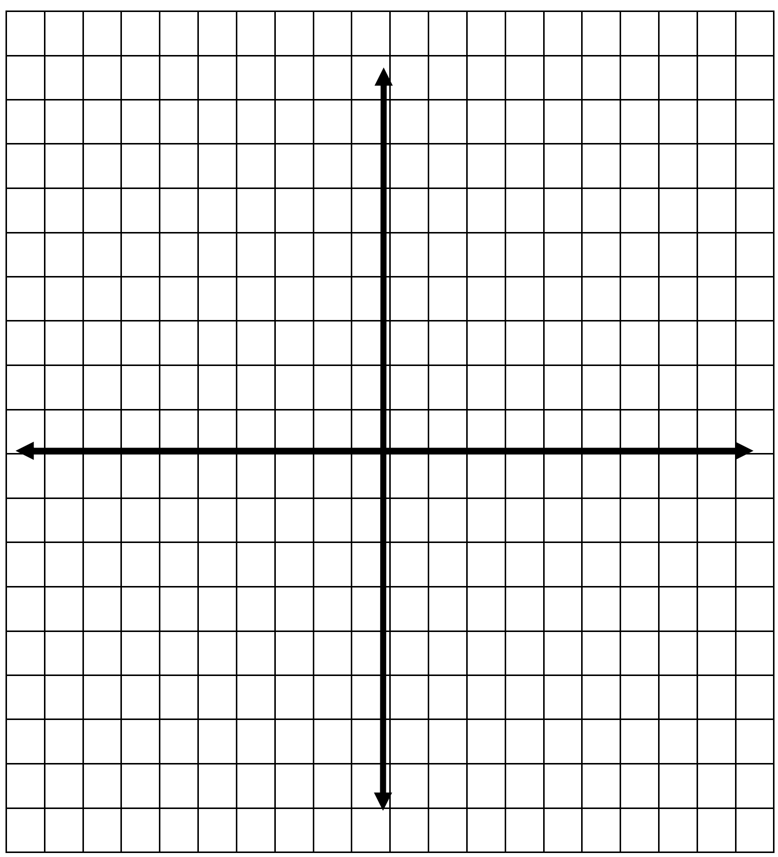
Rate of Change **AP CALCULUS AB** **Name ________________________**

1. **GIVEN: $P(x) = 2x^3 - 3x^2 + 4x - 5$**

   Using the difference quotient and h = 0, find the equation of the
   A) velocity function, and
   B) acceleration function, algebraically.

2. **Using the slope function and the table on your calculator, find the equation of the velocity function. Sketch the graphs of both functions below.**

   GIVEN: $P(x) = \ln(x)$



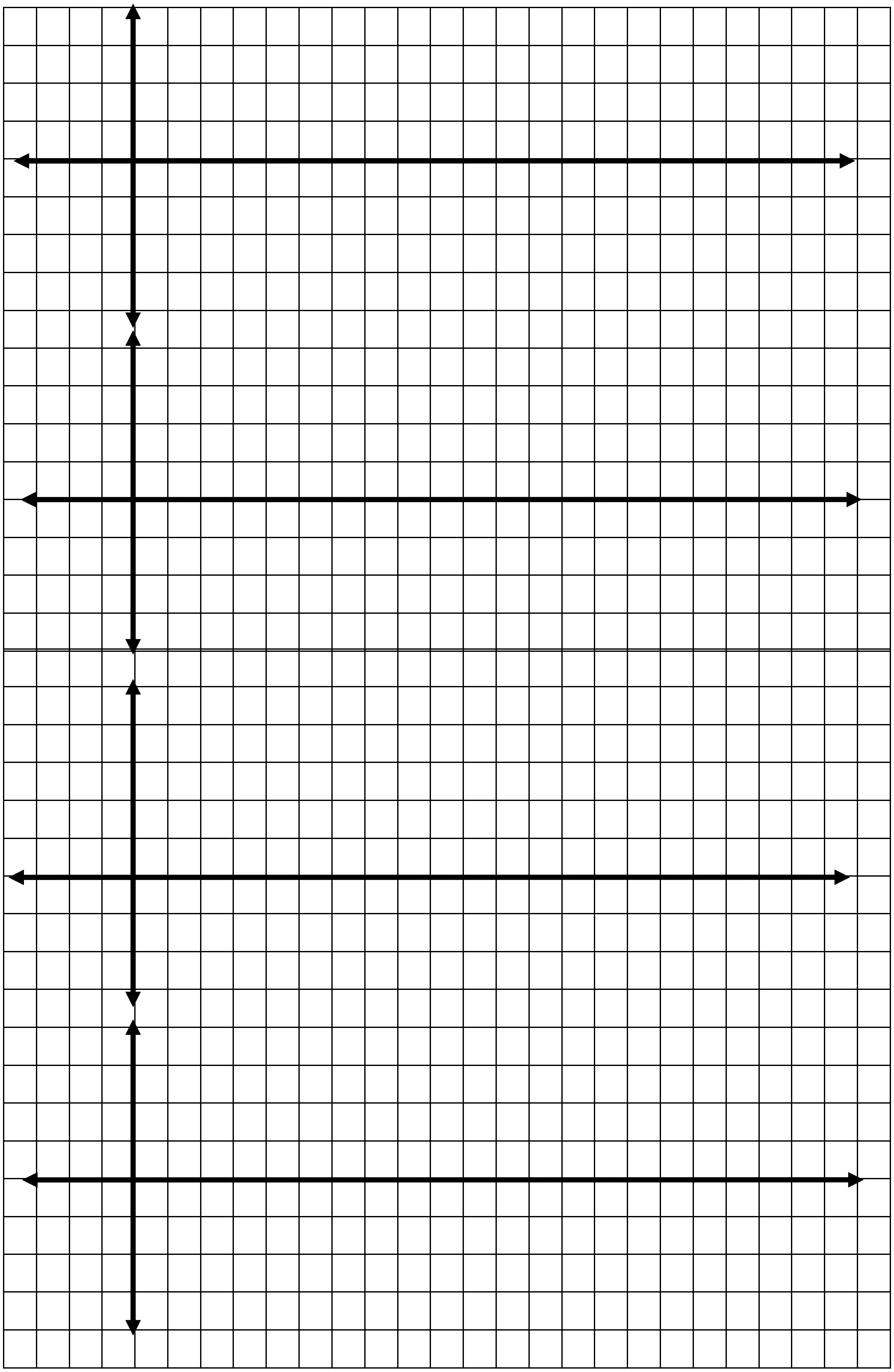
3. **GIVEN: P(x) = sin(x) Using the slope function on your calculator, find the equation of the A) velocity function,**

B) acceleration function, and

C) jerk function.

D) Sketch the graphs of <u>all four</u> functions below from $[0, 2\pi]$**.**

Use 6 blocks = π for the scale on the x-axis and 2 blocks = 1 for the y-axis.



P(x) = sin(x)

V(x) = ______________

A(x) = ______________

J(x) = ______________

Brick on a Spring **AP CALCULUS AB** **Name ____________________**

A brick is suspended on a spring and hangs at rest. The brick is pushed up a distance of 3 cm from its resting position. The brick is released at time t = 0 and allowed to oscillate.

1. Sketch a **diagram** of the brick bouncing to illustrate the ideal (never ending) situation.

2. The brick reaches its resting position after one second. Create a table that shows the position of the brick versus the time for the first 10 seconds in an ideal situation. Be sure to specify where the brick will be at zero seconds.

seconds	0	1	2	3	4	5	6	7	8	9	10
cm											

3. Create a graph (**on graph paper**) that models the motion of the brick in terms of time. Be sure to label the graph and axes.

Use your graph to answer questions 4-9.

4. Where is the function increasing? ______________________________

   Where is it decreasing? ______________________________

5. What are the maximum values of the function? _________________

   Where do they occur? _____________________ Label these points on the graph.

6. Interpret the maximum values in terms of the original physical situation.

7. What would be a reasonable domain for the function? _____________________

   What range would correspond to this domain? _____________________

8. Find an equation to describe this motion.

9. Would you expect the brick to oscillate forever in real life? Explain.

10. Suppose the brick is pushed up a distance of 6 cm from its resting position, and when released still reaches its resting position after one second.

a. Sketch a **diagram** illustrating the ideal situation.
b. Draw the graph (**on graph paper**).
c. Write the new **equation**. Be sure to label the graph and axes.
d. How does this change the graph? How does it change the equation?

seconds	0	1	2	3	4	5	6	7	8	9	10
cm											

11. Use the slope function on your calculator to find a graph of the rate of change for the brick's position function. Draw the rate graph (**on graph paper**) and find its **equation**.

12. Instead of being compressed, suppose the spring is pulled down so that the brick is again 3 cm from its resting position, and then released.

a. Sketch a **diagram** illustrating the ideal situation.
b. If the brick now reaches its resting position after one second, draw the graph (**on graph paper**). Be sure to label the graph and axes.
c. Write the new equation.

seconds	0	1	2	3	4	5	6	7	8	9	10
cm											

13. Suppose the spring is replaced with a less bouncy spring. The brick is pushed up a distance of 6 cm from its resting position, but now takes 2 seconds to reach its resting position.

a. Sketch a **diagram** illustrating the new situation.
b. If the brick now reaches its resting position after one second, draw the graph (**on graph paper**). Be sure to label the graph and axes.
c. Write the new equation.

seconds	0	2	4	6	8	10	12	14	16	18
cm										

14. Use the slope function on your calculator to find a graph of the rate of change for the brick's position function. Sketch the rate graph (**on graph paper**) and find its **equation**.

15. Compare the position and rate of change (slope) graphs from question 10 & 11 to those in questions 13 &14. State the changes and explain.

Algebraic expressions **AP CALCULUS AB** **Name ____________________**

Simplify completely. Show your work.

1. $\left(\dfrac{x^2+x}{x^2-4} \div \dfrac{x^2-1}{x^2+5x+6}\right) - \dfrac{4}{x^2+3x-4}$

2. $\dfrac{4-(1-w)^{-1}}{16+7(w^2-1)^{-1}}$

3. $\dfrac{25d^{-7/2}j^{8/3}}{15d^{3/2}j^{-2/3}}$

4. $\dfrac{\sqrt{x}-\sqrt{3}}{\sqrt{x}+\sqrt{3}}$

Solve for x. Check your answers.

5. $2x^{\frac{2}{3}}+7x^{\frac{1}{3}}-15=0$

a) y = sin (θ) if and only if θ = $\sin^{-1}$(y).
So, $f^{-1}(x) = \sin^{-1}(x) = \theta$ (the angle)

The domain of f(θ) = sin (θ) is all REAL numbers: $\theta \in \Re$.
The range of f(θ) is $-1 \le f(\theta) \le 1$.

The **domain** of the **<u>inverse</u>** function is $-1 \le x \le 1$.
The **range** of the **<u>inverse</u>** function is **limited.**

$$\frac{-\pi}{2} \le f^{-1}(x) \le \frac{\pi}{2} \quad \textbf{OR} \quad \frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$$

b) y = cos (θ) if and only if θ = $\cos^{-1}$(y).
So, $f^{-1}(x) = \cos^{-1}(x) = \theta$ (the angle)

The domain of f(θ) = cos (θ) is all REAL numbers: $\theta \in \Re$.
The range of f(θ) is $-1 \le f(\theta) \le 1$.

The **domain** of the **<u>inverse</u>** function is $-1 \le x \le 1$.
The **range** of the **<u>inverse</u>** function is **limited.**

$$0 \le f^{-1}(x) \le \pi \quad \textbf{OR} \quad 0 \le \theta \le \pi$$

c) y = tan (θ) if and only if θ = $\tan^{-1}$(y).
So, $f^{-1}(x) = \tan^{-1}(x) = \theta$ (the angle)

The domain of f(θ) = tan (θ)
is all REAL numbers excluding $\{\ldots -3\pi/2, -\pi/2, \pi/2, 3\pi/2, 5\pi/2, \ldots\}$ or
$\theta \ne (2k+1)\,\pi/2$ where k is an integer.
The range of f(θ) is all REAL numbers.

The **domain** of the **<u>inverse</u>** function is all REAL numbers.
The **range** of the **<u>inverse</u>** function is **limited.**

$$\frac{-\pi}{2} < f^{-1}(x) < \frac{\pi}{2} \quad \textbf{OR} \quad \frac{-\pi}{2} < \theta < \frac{\pi}{2}$$

Trig & Inverse Trig Functions **AP CALCULUS AB** **Name __________________**

1. Graph the following **on graph paper**, from $[-2\pi, 2\pi]$. Use 6 blocks = π for the scale on the x-axis and 2 blocks = 1 for the y-axis. Be sure to accurately plot common reference angle points (x = $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$, ...). Draw all <u>asymptotes</u>.

   a) f(x) = sin (x)

   b) g(x) = cos (x)

   c) h(x) = tan (x)

   d) k(x) = cot (x)

   e) n(x) = sec (x)

   f) p(x) = csc (x)

2. Use your calculator and draw the **inverse functions** on the <u>same</u> axes as the <u>original</u> functions (from question #1). Pay attention to the **<u>limited</u>** range of each. Draw all <u>asymptotes</u>.

   a) $y = \sin^{-1}(x)$

   b) $y = \cos^{-1}(x)$

   c) $y = \tan^{-1}(x)$

3. Use the **unit circle** to give the angle measure of each trigonometric expression. Give your answer in **radian** measure using π (no calculator). Remember the quadrants (range) for which each **inverse** function is defined.

a) $\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)$ = ______

b) $\sin^{-1}(-1)$ = ______

c) $\cos^{-1}\left(\frac{1}{2}\right)$ = ______

d) $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ = ______

e) $\cos^{-1}(0)$ = ______

f) $\sin^{-1}(1)$ = ______

m) $\cos^{-1}(-1)$ = ______

n) $\sin^{-1}\left(-\frac{1}{2}\right)$ = ______

o) $\tan^{-1}(-1)$ = ______

p) $\tan^{-1}(-\sqrt{3})$ = _____

q) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ = ______

g) $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$ = ______

h) $\cos^{-1}(1)$ = ______

i) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ = ______

j) $\cos^{-1}\left(-\frac{1}{2}\right)$ = ______

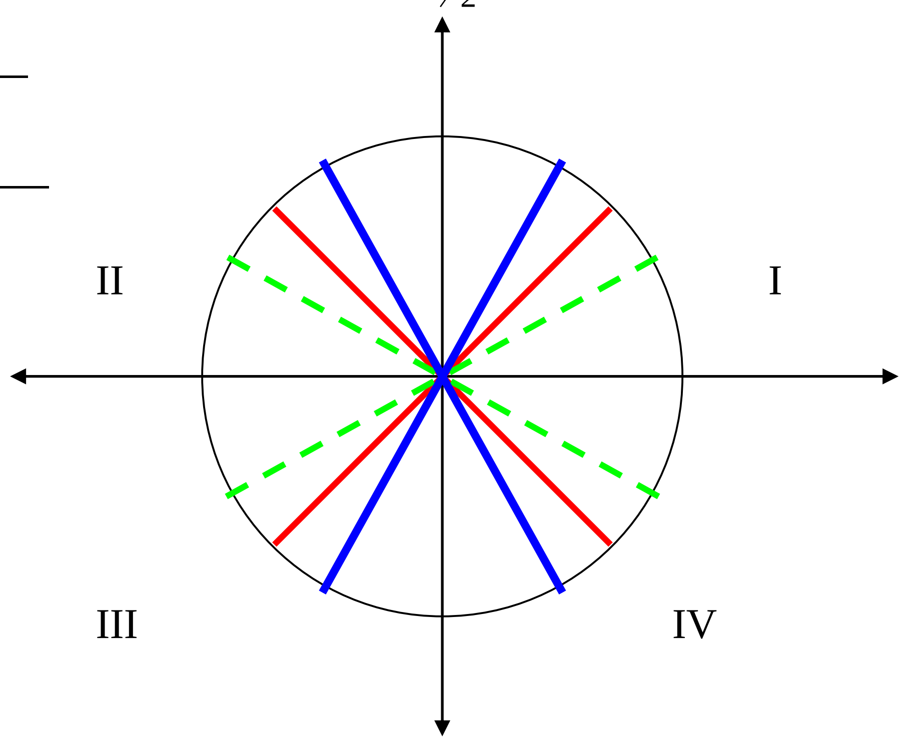
k) $\tan^{-1}(\sqrt{3})$ = ______

l) $\tan^{-1}(1)$ = ______

r) $\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$ = ______

s) $\tan^{-1}(1)$ = ______

t) $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ = ______



4. A. If $\cos^{-1}\left(-\frac{7}{25}\right) = \theta$ in Quadrant II, find:

B. If $\tan^{-1}(\sqrt{6}) = \alpha$ in Quadrant I, find:

(No calculator. Hint: Draw a right triangle for each.)

a) $\sin(\theta)$ = ______________________

b) $\cos(\theta)$ = ______________________

c) $\tan(\theta)$ = ______________________

d) $\cot(\theta)$ = ______________________

e) $\sec(\theta)$ = ______________________

f) $\csc(\theta)$ = ______________________

a) $\sin(\alpha)$ = ______________________

b) $\cos(\alpha)$ = ______________________

c) $\tan(\alpha)$ = ______________________

d) $\cot(\alpha)$ = ______________________

e) $\sec(\alpha)$ = ______________________

f) $\csc(\alpha)$ = ______________________

5. Using the information (a)(b) = 0 if and only if a = 0 or b = 0, solve the following equation in the interval $[0, 2\pi)$. (No calculator.)

$$2\cos(x)\sin(x) - \sin(x) = 0$$

6. Any point on the **unit circle** has the coordinates $(\cos(\theta), \sin(\theta))$, so $x = \cos(\theta)$ and $y = \sin(\theta)$.

From the Pythagorean theorem, $x^2 + y^2 = r^2$ and substituting for x and y, then **$\sin^2(\theta) + \cos^2(\theta) = 1$.**

Using the following identities, solve the equations for x in the interval $[0, 2\pi)$.
(No calculator.)

Pythagorean Identity: **$\sin^2(x) + \cos^2(x) = 1$ → $1 - \sin^2(x) = \cos^2(x)$ OR $1 - \cos^2(x) = \sin^2(x)$**

a. Solve for x, $[0, 2\pi)$: $\sin^2(x) - \cos^2(x) = 0$

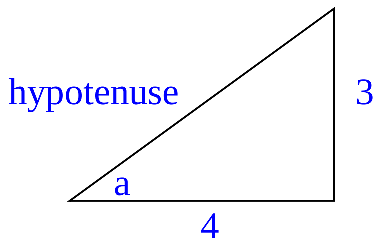
b. Solve for x, $[0, 2\pi)$: $2\sin(x)\tan(x) + \tan(x) - 2\sin(x) - 1 = 0$ (factor by grouping)

Trigonometry AP CALCULUS AB Name ____________________

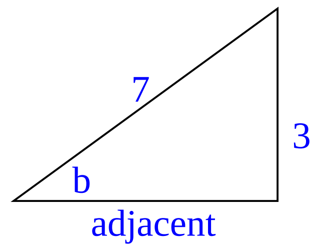
Use the formulas given on the next page to calculate the value of the given expressions on the following pages **exactly** (no calculator). Follow the examples.

GIVEN: tan(a) = ¾ and csc(b) = 7/3, in Quadrant I, find cos(a – b), tan(2b) & cos(½a).

Using a right triangle (or the Pythagorean Identity), find the values of the other trig functions.



$hyp^2 = 3^2 + 4^2$
hyp = 5



$3^2 + adj^2 = 7^2$
$adj = 2\sqrt{10}$

From the first triangle, $\sin(a) = \frac{3}{5}$ and $\cos(a) = \frac{4}{5}$ and $\tan(a) = \frac{3}{4}$.

From the second triangle, $\sin(b) = \frac{3}{7}$ and $\cos(b) = \frac{2\sqrt{10}}{7}$ and $\tan(b) = \frac{3\sqrt{10}}{20}$.

<u>FIND</u>: cos(a – b), using the formula, cos(a – b) = cos(a)cos(b) + sin(a) sin(b).

Substitute into the formula and simplify.

$$\cos(a - b) = \cos(a)\cos(b) + \sin(a)\sin(b)$$
$$= (4/5)(2\sqrt{10}/7) + (3/5)(3/7)$$
$$= (8\sqrt{10} + 9)/35$$

<u>FIND</u>: tan (2a), using the formula, $\tan(2a) = \frac{2\tan(a)}{1-\tan^2(a)}$.

$$\tan(2a) = \frac{2\tan(a)}{1-\tan^2(a)}$$
$$= \frac{2(3/4)}{1-(3/4)^2} = \frac{24}{7}$$

<u>FIND</u>: cos(½b), using the formula $\cos(\tfrac{1}{2}b) = \pm\sqrt{\frac{1+\cos(b)}{2}}$.

$$\cos(\tfrac{1}{2}b) = \sqrt{\frac{1+\frac{2\sqrt{10}}{7}}{2}} = \sqrt{\frac{7+2\sqrt{10}}{14}}$$ (Okay to leave in this form.)

FORMULAS for sum & difference of angles, double angle and half-angle

sin(a + b) = sin(a)cos(b) + cos(a) sin(b)

sin(a – b) = sin(a)cos(b) – cos(a) sin(b)

cos(a + b) = cos(a)cos(b) – sin(a) sin(b)

cos(a – b) = cos(a)cos(b) + sin(a) sin(b)

$\tan(a + b) = \frac{\tan(a) + \tan(b)}{1 - \tan(a)\tan(b)}$

$\tan(a - b) = \frac{\tan(a) - \tan(b)}{1 + \tan(a)\tan(b)}$

sin(2a) = 2sin(a) cos(a)

$\cos(2a) = \cos^2(a) - \sin^2(a)$

$\tan(2a) = \frac{2\tan(a)}{1 - \tan^2(a)}$

$\sin(\tfrac{1}{2}a) = \pm\sqrt{\frac{1 - \cos(a)}{2}}$

$\cos(\tfrac{1}{2}a) = \pm\sqrt{\frac{1 + \cos(a)}{2}}$

$\tan(\tfrac{1}{2}a) = \pm\sqrt{\frac{1 - \cos(a)}{1 + \cos(a)}}$

(The sign is determined by the quadrant.)

OR $\sin^2(a) = \frac{1 - \cos(2a)}{2}$ $\cos^2(a) = \frac{1 + \cos(2a)}{2}$ (Note: These are still double angle formulas.)

The ones in red are used most often in AP Calculus.

http://www.themathpage.com/atrig/trigonometric-identities.htm#double
Use this link to find the proof and explanation of the trigonometric identities above.

GIVEN: tan(a) = $\frac{2}{5}$ in Quadrant III and cos(b) = $-\frac{4}{9}$ in Quadrant II
Set up the right triangles for $\angle a$ and $\angle b$. Find the lengths of the missing side. This information will be used for the problems on the next page.

Use the triangles from the bottom of the previous page to find:

1. $\sin(a - b)$

5. $\sin(2a)$

2. $\cos(a + b)$

6. $\cos(\frac{1}{2}b)$

3. $\tan(a - b)$

7. $\sin^2(c)$, if $\cos(2c) = \frac{1}{3}$

4. $\cos(2b)$

8. $\cos^2(c)$, if $\cos(2c) = \frac{1}{3}$

Parent Functions & Transformations

Name ____________________

Create a deck of cards for matching. You will need to know these graphs.

A) Write the parent function (column 1 below) on the first card.
B) State its domain and range on a second card.
C) Draw its graph on a third card. (You can get index cards with a grid on them if you want!)
D) Write the transformation described (column 2 below) on a fourth card.
E) Draw the graph for the transformation described on a fifth card.
F) Write the equation for the transformation described on a sixth card.

You will need 84 cards total. **Use a different color for each group.**

Parent Functions	Transformations
1. $f(x) = x$	Vertical shrink of ⅓.
2. $f(x) = \|x\|$	Horizontal shift right 2, vertical stretch of 3
3. $f(x) = [\![x]\!]$	Vertical shift up 1
4. $f(x) = x^2$	Vertical stretch of 2
5. $f(x) = x^3$	Horizontal shift left 3
6. $f(x) = \frac{1}{x}$	Horizontal shift left 2, vertical shift down 1
7. $f(x) = \sqrt{x}$	Horizontal shift left 3
8. $f(x) = e^x$	Vertical shrink of ½
9. $f(x) = \frac{1}{2}^x$	Vertical stretch of 4, vertical shift up 1
10. $f(x) = \ln(x)$	Horizontal shift left 3
11. $f(x) = \sin(x)$ from $[0, 2\pi]$	Horizontal shrink of 3
12. $f(x) = \cos(x)$ from $[0, 2\pi]$	Horizontal stretch of 2
13. $f(x) = \tan(x)$ from $[0, 2\pi]$	Reflection over the x axis
14. $x^2 + y^2 = 1$	Horizontal shift right 1, vertical shift down 1